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An investigation of stability of a control surface with structural nonlinearities in supersonic flow using Zubov's method

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Abstract

It is well known that the presence of nonlinearities may significantly affect the aeroelastic response of an aerospace vehicle structure. In this paper, the aeroelastic behaviour at high Mach numbers of an all-moving control surface with a nonlinearity in the root support is investigated. Very often, under certain flight conditions, a stable equilibrium point, corresponding to zero displacement of the structure, together with an unstable limit cycle arising from a sub-critical Hopf bifurcation results from the presence of the nonlinearity. The dynamic aeroelastic response to external excitation is also of interest, and when sinusoidal forcing is applied, the stable equilibrium point may then be replaced by a periodic attractor, and the limit cycle by an unstable multi-periodic solution. With or without this forcing, there is an attractor which will possess a domain of attraction. In this paper, the problem of estimating these domains of attraction is tackled using Zubov's method. In the absence of forcing, the method is applied directly to the aeroelastic equations, while for the forced system, the method of averaging is applied to approximate the aeroelastic equations by an autonomous system. The behaviour of the system with forcing is also investigated for flight speeds below a threshold which may occur where the unstable limit cycle of the unforced system disappears. In this regime, the nonlinear system may nevertheless still possess multiple attractors, and their domains of attraction are investigated, again using an averaged form of the aeroelastic equations. In this study, the nonlinearity in the root support was assumed to be due to a cubic hardening restoring moment. The Zubov approach, which always yields conservative estimates, was shown to be capable of rapidly giving a good indication of stability domain boundaries under many conditions. Although this investigation focuses on an aeroelastic system, the general form of equations considered arises in many other settings, so that the approach would be relevant to a whole range of engineering applications.

1. Introduction

In carrying out aeroelastic analysis for an aircraft or missile, the structure, aerodynamics and controls (if considered) are generally modelled with the assumption of linearity. A major objective of the analysis is to determine a flutter boundary in terms of flight speed and various design parameters. In practice, nonlinearities may be present that are capable of significantly affecting aeroelastic behaviour. Typical phenomena resulting

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Nomenclature

A, B	matrices in averaged forms of aeroelastic equations using invertible van der Pol transformation
a_1, a_2, a_3, a_4	slowly varying variables in averaging procedure
a_i^F	*i*th generalised mass of control surface
a_∞	free stream speed of sound
b	control surface root semi-chord
b_s	mean semi-chord for *s*th strip of control surface
e_i^F	*i*th generalised stiffness of control surface
d_{ij}^F	structural damping coefficient
f	function defining nonlinearity in fin root support
f_i	right hand side of *i*th nonlinear equation
G	damping matrix in aeroelastic equations
H	stiffness matrix in aeroelastic equations
h_s	vertical displacement of *s*th strip at point O_s
I_{ij}	coefficient of $\dot{X}_j$ in equation for generalised aerodynamic force Q_{Ai}
$I_{1s},\ldots, I_{5s}$	integrals depending on control surface thickness used in aerodynamics
K	linear torsional stiffness at root
L_s	load per unit span for *s*th strip
$\bar{\mathbf{M}}$	right hand modal matrix in solution of linearised aeroelastic equations
$\bar{\mathbf{N}}$	left hand modal matrix in solution of linearised aeroelastic equations
M_∞	free stream Mach number
M_s	torsional moment per unit span for *s*th strip about O_s
P	transformation matrix used in averaging aeroelastic equations
$\mathbf{p}_0$, $\mathbf{q}_0$	transformed $\mathbf{x}_0$, $\mathbf{y}_0$
Q_1, Q_2, Q_3, Q_4	generalised coordinates arising from transforming **X** and **Y**
Q'_1, Q'_2, Q'_3, Q'_4	transformed generalised coordinates used in averaged form of aeroelastic equations
Q_{Ai}	*i*th generalised aerodynamic force
Q_{Ei}	*i*th generalised force due to external excitation
Q_{Mi}	*i*th generalised force on fin due to torsional reaction loads
R_{ij}	coefficient of X_j in equation for generalised aerodynamic force Q_{Ai}
r	column vector relating fin root torsion angle to generalised displacements in aeroelastic equations
s	column vector used in defining nonlinear terms in aeroelastic equations
t	time
t'	non-dimensionalised time ($=\tilde{\omega}t$)
U_∞	free stream velocity
V_∞	non-dimensionalised speed in aeroelastic equations
V_n	*n*th order term in series expansion for V
$V^{(N)}$	series expansion of V up to order N
V	Lyapunov function
$x_{\rm i}$	*i*th dependent variable in nonlinear equations
x_s, y_s, z_s	coordinates for *s*th strip in aerodynamic model
x_{os}	non-dimensional distance of O_s from leading edge of *s*th strip
X_1, X_2	generalised displacements
$\mathbf{x}_0$, $\mathbf{y}_0$	amplitudes of external forcing function
X	column vector of generalised displacements
Y	derivative of **X** with respect to time
Z	transformation of **X** in averaged form of aeroelastic equations using invertible van der Pol method
w	downwash on control surface
γ	ratio of specific heats
Δh	strip width
ε	nonlinearity parameter
θ_s	pitch rotation of *s*th strip
θ_y	pitch rotation of control surface at root
ρ_∞	free stream density
$\tau_s(x_s)$	non-dimensional thickness distribution function for *s*th strip
ϕ	positive definite function in Zubov equation
ϕ_i^F	*i*th natural mode for control surface
ϕ_{is}^F	*i*th modal displacement of the *s*th strip
ψ_{iys}^F	*i*th modal pitch rotation of the *s*th strip
ψ_{iy0}^F	*i*th modal pitch rotation of control surface at root
v	alternative form of Lyapunov function
$\tilde{\omega}$	frequency used in non-dimensionalising time t
ω_i^F	*i*th natural frequency of control surface
ω_1, ω_2	frequencies in solution of linear aeroelastic system
ω_0	forcing frequency

from the presence of nonlinearities include the onset of stable limit cycle oscillations through a super-critical Hopf bifurcation beyond the flutter boundary determined by linear theory, or the existence of unstable limit cycles within the linear flutter boundary associated with a sub-critical Hopf bifurcation. In the former case, the effect of the nonlinearity may be regarded as beneficial if the limit cycle oscillations are small, whereas in the latter case, the nonlinearities could lead to the possibility of divergent oscillations occurring within the linear flutter boundary as the equilibrium point of the system will possess some domain of attraction. It is this case that is considered in this paper. The aeroelastic behaviour in response to external excitation is also of interest, and when sinusoidal forcing is applied, the stable equilibrium point may then be replaced by a periodic attractor which will also possess a domain of attraction. The behaviour of the system with forcing is of interest for flight speeds below a threshold where the unstable limit cycle of the unforced system disappears. In particular, close to a resonance, the nonlinear system may exhibit multiple attractors, which will therefore possess domains of attraction.

One approach to carry out a theoretical study of nonlinear aeroelastic behaviour is to perform the analysis in the time domain. However, a drawback with this is that though it can yield a complete picture of system behaviour for a particular set of initial conditions, it may be inefficient in providing an overall picture of system characteristics even for a single set of system parameters. In aeroelastic studies carried out during development of an aerospace vehicle, it is necessary to consider a wide range of flight conditions and design parameters, and thus there is a strong motivation to apply or develop alternative analysis techniques. Amongst these are averaging methods [1–10] in some of which, nonlinearities are replaced by 'equivalent' stiffnesses or dampings. These are attractive as they may then enable linear analysis techniques to be applied. A wide range of bifurcational behaviour may be encountered in nonlinear aeroelastic systems, and a number of investigations have demonstrated this both theoretically and experimentally [11–17]. Consequently, the application of nonlinear dynamical systems theory in the field of aeroelasticity is of great interest. Holmes [18] investigated panel flutter in terms of the bifurcations that are possible as in-plane load and air speed are varied, while Anderson employed generic modelling to interpret observed nonlinear transonic aeroelastic behaviour and to suggest the existence of new phenomena not previously encountered in computational studies [19]. Dowell et al. [12] studied conditions necessary for chaotic motion of a buckled plate with external excitation in an aerodynamic flow. Centre manifold theory has been used to predict limit cycle oscillations in nonlinear aeroelastic systems. Examples of this approach include Grzedzinski [20], Liu et al. [21] and Sedaghat et al. [22].

Many applications of the analysis techniques discussed above have been concerned with the investigation of possible limit cycles of nonlinear aeroelastic systems. This paper considers an all-moving control surface flying under conditions where it possesses attractors with domains of attraction. In this paper, analysis is carried out in terms of estimating these domains of attractions, and thus focuses on a slightly different aspect of the nonlinear behaviour of aeroelastic systems. A variety of techniques have been developed to tackle the problem of stability domain estimation. One approach is that of Lee et al. [23] who used the interpolated mapping method. Another approach is through the determination of the stable manifold for unstable equilibria which was taken by Lewis [24], who investigated a second-order two-degree-of-freedom (dof) nonlinear aeroelastic system.

Another major class of methods is those based on the Lyapunov method, and in particular the construction method of Zubov [25]. He showed that Lyapunov functions giving the entire domain of stability of an autonomous nonlinear system satisfy a certain partial differential equation. Many papers have discussed how approximate solutions to Zubov's equation, and hence domain of attraction estimates, can be obtained in practice. Approaches include the use of truncated power series approximations [26,27] and Lie series based methods [28,29]. Many illustrations of the use of the method are limited to two-dof first-order systems, although one example of analysis of a four-dof system is due to Dimantha et al. [27]. The aeroelastic equations considered here also comprise a four-dof first-order system. This paper shows the practicality of Zubov's method for the stability domain analysis of such a system by way of the power series approach implemented through the use of computer algebra. It builds on the study by Lewis [24] through exploring another approach to stability domain analysis and by extending the investigation to consider the frequency response of the system.

Although formulations of Zubov's equation for certain classes of non-autonomous systems, or autonomous systems with periodic attractors have been developed [30,31], the determination of stability domains in practice does not appear straightforward. Hence, in studying the forced nonlinear aeroelastic system, the averaging approach was adopted to obtain an approximate autonomous system of equations. This was done for both resonance and non-resonance conditions. The use of the method of averaging in stability domain estimation has been considered by Loud and Sethna [32] and Gilsinn [33]. Gilsinn showed that one might anticipate that the use of an averaged system of equations would lead to conservative stability domain estimates. This study therefore brings together two nonlinear analysis techniques, namely perturbation methods and Lyapunov theory, to tackle the problem of stability domain estimation with the aid of computer algebra in a practical manner, suitable for parametric studies.

Although this investigation focuses on an aeroelastic system, it is important to note that the general form of governing equations for the system considered arise in many settings, so that the approach described in this paper would be applicable in a range of engineering scenarios.

The layout of this paper is as follows. Section 2 discusses the derivation of the aeroelastic equations for the all-moving control surface in high speed supersonic flow. In Section 3, averaged forms of the equations are obtained. Section 4 then reviews Zubov's approach to stability domain determination and practical approaches to obtaining stability domain boundaries. Section 5 presents results from Zubov's method and comparisons are made with predictions of the domains of attraction of the attractors of the system obtained by numerical integration of the aeroelastic equations in their original form. Throughout, the case of a cubic-hardening nonlinearity in the torsional dof of the root support of the control surface is considered. Concluding remarks are then given in Section 6.

2. Aeroelastic equations

In this section, the form of the aeroelastic equations for an all-moving control surface with a single root support in a uniform supersonic flow is discussed. It is assumed that the control surface structure deforms linearly, and that the root support force-deflection characteristics are linear in bending but nonlinear in torsion. The control surface is assumed to be performing small oscillations about a mean incidence angle of zero. The equations of motion for the control surface subjected to a set of external forces are first derived. These external forces comprise the aerodynamic loads obtained using third-order piston theory [34], the nonlinear torsional moment about the control surface hinge line due to the root support, and external excitation.

The undisplaced control surface is illustrated in Fig. 1. Fixed Cartesian axes are defined with the origin P being the point where the root support is located. Control surface chordwise sections are assumed to be symmetrical, with the x-axis taken to be along the root chord line of symmetry with the positive direction towards the trailing edge; the y-axis is then the hinge line for the undeformed control surface, for which the x–y plane is a plane of symmetry. If the control surface is modelled as a plate structure, then its deformation may be defined in terms of vertical displacements of the mid-surface and rotations about the x and y axes.

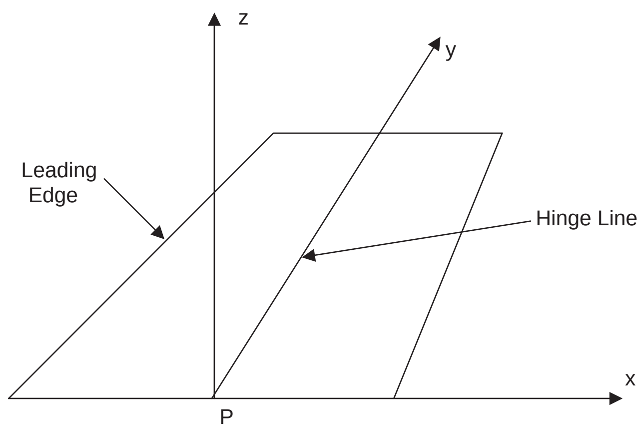

Fig. 1. Lifting surface and axis system.

The control surface root chord is assumed to be unconstrained except at the root support P where bending and torsional rotations may occur, but no translational displacements are permitted. Modes and frequencies for the control surface are then determined for a given bending stiffness and an assumed linear torsional stiffness of the root support. Let the control surface modes be $\phi_1^F(x,y)$, $\phi_2^F(x,y),\ldots$ and let the associated natural frequencies be ω_1^F, $\omega_2^F\ldots$. The generalised masses and stiffnesses of the control surface may then be evaluated and are denoted by a_1^F, $a_2^F\ldots$ and e_1^F, $e_2^F\ldots$, respectively.

Suppose now that the control surface is subject to forces through aerodynamic loadings arising from structural deformation, torsional reaction loads due to the nonlinearity in the root support, together with an external forcing function. Then the equations of motion for the control surface may be obtained in the form:

$$a_i^F \ddot{X}_i + \sum_j d_{ij}^F \dot{X}_j + e_i^F X_i = Q_{Mi} + Q_{Ai} + Q_{Ei} \tag{1}$$

for $i = 1, 2, 3,\ldots$ where d_{ij}^F denotes a structural damping coefficient and X_1, X_2, … are generalised coordinates defining the control surface motion in terms of the modes $\phi_1^F(x,y)$, $\phi_2^F(x,y)$, …. Q_{Mi} is the ith generalised force due to the torsional reaction loads exerted on the control surface by the nonlinearity in the root support, Q_{Ai} is the ith generalised force due to the aerodynamic loads, Q_{Ei} is the ith generalised force due to external forcing.

Let M_y be the torsional moment acting on the control surface due to the nonlinearity in the root support. Assuming that this is a nonlinear function of the control surface root torsional angle θ_y and angular rate $\dot{\theta}_y$, Q_{Mi} is given by:

$$Q_{Mi} = M_y(\theta_y, \dot{\theta}_y)\psi_{iy0}^F \tag{2}$$

where ψ_{iy0}^F is the torsional rotation of the control surface at the root for the ith mode so that θ_y may be written:

$$\theta_y = \sum_i \psi_{iy0}^F X_i \tag{3}$$

In this investigation, the aerodynamic loadings on the control surface are evaluated using third-order piston theory [34] applied as strip theory so that the control surface is divided spanwise into a series of aerofoils undergoing heaving and pitching motions for which the lift and pitching moment may readily be determined. This approach is justifiable so long as the control surface does not have a very low aspect ratio or where significant chordwise deformations occur. Piston theory aerodynamics is applied at high supersonic speeds typically corresponding to Mach numbers between 2.5 and 7.0. The key assumption in piston theory is that the local pressure on an aerofoil surface is related to the normal component of the local fluid velocity in the same manner that the pressure on a piston in a fluid filled tube is related to the velocity of the piston. In the form of the theory used for the present study, the pressure at a point on the fin surface is given by:

$$p - p_\infty = \rho_\infty a_\infty^2 \left\{ \frac{w}{a_\infty} + \frac{1}{4}(\gamma+1)\left(\frac{w}{a_\infty}\right)^2 + \frac{1}{12}(\gamma+1)\left(\frac{w}{a_\infty}\right)^3 \right\} \tag{4}$$

where w is the local downwash for the moving lifting surface, ρ_∞ is the free stream density, a_∞ the freestream speed of sound, p_∞ freestream pressure and γ the ratio of specific heats. In general, the downwash may be written as a sum of steady state contributions due to thickness and mean angle of attack of the lifting surface together with unsteady contributions arising from its motion. In this study, a zero mean incidence is assumed, and only terms linear in unsteady displacements are retained.

Consider now the symmetric aerofoil section shown in Fig. 2, having semi-chord b_s. In the undisturbed position, it lies with its line of symmetry along the $O_s x_s$ axis of a coordinate system where x_s, y_s, z_s are non-dimensionalised with respect to $2b_s$, with the leading edge a distance x_{os} forward of the origin O_s. The aerofoil thickness distribution, non-dimensionalised with respect to $2b_s$, is given by $\tau_s(x_s)$. The aerofoil lies in a uniform flow with velocity U_∞. and undergoes heaving and pitching motions h_s and θ_s as defined in Fig. 2. Thus the downwash on the aerofoil is given by:

$$\frac{w}{U_\infty} = \pm\left[\frac{\dot{h}_s}{U_\infty} + \theta_s + \frac{2b_s}{U_\infty}(x_s - x_{0s})\dot{\theta}_s\right] + \frac{1}{2}\frac{\mathrm{d}\tau_s}{\mathrm{d}x_s} \tag{5}$$

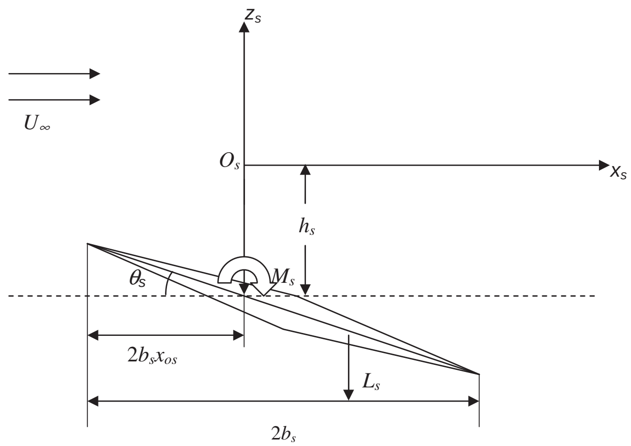

Fig. 2. Notation for analysis of an aerofoil section.

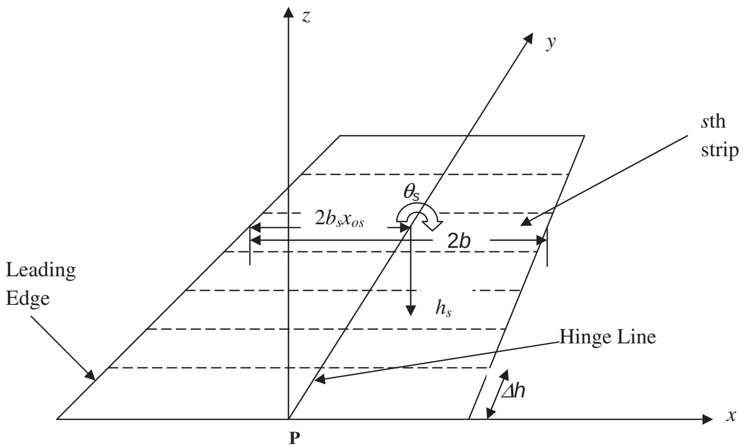

Fig. 3. Notation used in implementation of strip theory.

where the minus sign applies to the upper surface and the plus sign to the lower surface of the aerofoil. Substituting Eq. (5) into Eq. (4) will then enable the pressure difference between the upper and lower aerofoil surfaces to be obtained. The force per unit span L_s and the pitching moment per unit span M_s about O_s defined in the sense shown in Fig. 2 may then be obtained. Given these, the aerodynamic loading on the lifting surface may now be determined by strip theory. Let the lifting surface be divided up into strips, capable of vertical translation and pitch rotation due to its elastic deformation as shown in Fig. 3. h_s and θ_s are now the displacement and pitch rotation for the sth strip as shown. The average distance of the leading edge of the sth strip from the y_s axis is $2b_s x_{os}$. Let b be the length of the root semi-chord. L_s and M_s are then given by:

$$L_s = -4\rho_\infty b U_\infty^2 \left\{ \frac{\lambda_{s1}}{U_\infty M_\infty} \dot{h}_s + \frac{\lambda_{s2}}{M_\infty} \theta_s + \frac{\lambda_{s3}}{U_\infty M_\infty} \dot{\theta}_s \right\}$$

$$M_s = -4\rho_\infty b U_\infty^2 \left\{ \frac{\mu_{s1}}{U_\infty M_\infty} \dot{h}_s + \frac{\mu_{s2}}{M_\infty} \theta_s + \frac{\mu_{s3}}{U_\infty M_\infty} \dot{\theta}_s \right\} \tag{6}$$

where

$$\lambda_{s1} = \lambda_{s2} = \frac{b_s}{b}\left(1 + \frac{1}{4}(\gamma+1)M_\infty^2 I_{3s}\right)$$

$$\lambda_{s3} = \frac{b_s^2}{b}\left\{1 - 2x_{os} + (\gamma+1)M_\infty I_{1s} + \frac{1}{2}(\gamma+1)M_\infty^2(I_{4s} - x_{os}I_{3s})\right\}$$

$$\mu_{s1} = \mu_{s2} = \lambda_{s3}$$

$$\mu_{s3} = \frac{b_s^3}{b}\left\{\frac{4}{3} - 4x_{os} + 4x_{os}^2 + 2(\gamma+1)(I_{2s} - 2x_{os}I_{1s})M_\infty + (\gamma+1)M_\infty^2\left(I_{5s} - 2x_{os}I_{4s} + x_{os}^2 I_{3s}\right)\right\} \tag{7}$$

and I_{1s}, I_{2s}, I_{3s}, I_{4s} and I_{5s} are functions of thickness distribution and are given by:

$$I_{1s} = \frac{1}{2}\int_0^1 x_s \frac{d\tau_s}{dx_s}\,dx_s; \quad I_{2s} = \frac{1}{2}\int_0^1 x_s^2 \frac{d\tau_s}{dx_s}\,dx_s; \quad I_{3s} = \frac{1}{4}\int_0^1 \left(\frac{d\tau_s}{dx_s}\right)^2 dx_s;$$

$$I_{4s} = \frac{1}{4}\int_0^1 x_s\left(\frac{d\tau_s}{dx_s}\right)^2 dx_s; \quad I_{5s} = \frac{1}{4}\int_0^1 x_s^2\left(\frac{d\tau_s}{dx_s}\right)^2 dx_s \tag{8}$$

where it is assumed that the leading and trailing edge thicknesses are zero so that $\tau_s(0) = \tau_s(1) = 0$. Denote by ϕ_{is}^F and ψ_{iys}^F the ith modal displacement and rotation of the sth strip. Then h_s and θ_s may be written in terms of generalised displacements as:

$$h_s = -\sum_i \phi_{is}^F X_i$$

$$\theta_s = \sum_i \psi_{iys}^F X_i \tag{9}$$

The virtual work per unit span on the sth strip by the aerodynamic forces in producing a translation δh_s and pitch rotation $\delta\theta_s$ as a result of increments δX_1, $\delta X_2, \ldots$ in the generalised displacements may then be determined, from which the generalised aerodynamic force for the sth strip, Q_{si}^A, can be found from:

$$Q_{si}^A = \Delta h(M_s\psi_{iys}^F - L_s\phi_{is}^F) \tag{10}$$

where Δh is the strip width. Making use of Eqs. (6)–(9) in Eq. (10) then leads to the following expression for Q_{si}^A:

$$Q_{si}^A = -4\rho_\infty U_\infty^2 b\Delta h\sum_j\left[\frac{A_{ij}^s}{M_\infty}X_j + \frac{B_{ij}^s}{M_\infty U_\infty}\dot{X}_j\right] \tag{11}$$

where A_{ij}^s and B_{ij}^s are given by:

$$A_{ij}^s = \mu_{s2}\psi_{iys}^F\psi_{jys}^F - \lambda_{s2}\phi_{is}^F\psi_{jys}^F$$

$$B_{ij}^s = -\mu_{s3}\psi_{iys}^F\psi_{jys}^F - \mu_{s1}\phi_{js}^F\psi_{iys}^F + \lambda_{s1}\phi_{is}^F\phi_{js}^F + \lambda_{s3}\psi_{jys}^F\phi_{is}^F \tag{12}$$

The ith generalised force for the overall aerodynamic loading on the lifting surface is then obtained by summing the contributions for all strips. Thus writing:

$$R_{ij} = \Delta h\sum_s A_{ij}^s; \quad I_{ij} = \Delta h\sum_s B_{ij}^s \tag{13}$$

enables the generalised force Q_{Ai} due to the aerodynamic forces to be written in terms of the generalised displacements X_1, X_2, $X_3, \ldots$ in the following way:

$$Q_{Ai} = -4\rho_\infty U_\infty^2 b\sum_j\left[\frac{R_{ij}}{M_\infty}X_j + \frac{I_{ij}}{M_\infty U_\infty}\dot{X}_j\right] \tag{14}$$

The external forcing function will be assumed to be sinusoidal with frequency ω_0. Thus combining Eqs. (1), (2) and (14) together with the external forcing term then yields the aeroelastic equations in the form:

$$\ddot{\mathbf{X}} + \mathbf{G}\dot{\mathbf{X}} + \mathbf{H}\mathbf{X} + \varepsilon f(\mathbf{r}^{\mathbf{T}}\mathbf{X}, \mathbf{r}^{\mathbf{T}}\dot{\mathbf{X}})\mathbf{s} = \mathbf{x}_0 \sin \omega_0 t + \mathbf{y}_0 \cos \omega_0 t \tag{15}$$

where $\mathbf{X}$ is the column vector of the generalised coordinates $X_1, X_2, \ldots, \varepsilon$ is a parameter governing the strength of the nonlinearity. $\mathbf{G}$ and $\mathbf{H}$ are damping and stiffness matrices whose elements are given by:

$$G_{ij} = \frac{d_{ij}^F}{a_i^F}\delta_{ij} + \frac{4\rho_\infty b U_\infty^2 I_{ij}}{M_\infty U_\infty a_i^F}$$

$$H_{ij} = \frac{e_i^F}{a_i^F}\delta_{ij} + \frac{4\rho_\infty b U_\infty^2 R_{ij}}{M_\infty a_i^F} \tag{16}$$

and $\mathbf{r}$, $\mathbf{s}$ are column vectors whose elements are given by:

$$r_i = \psi_{iy0}^F; \quad s_i = \frac{\psi_{iy0}^F}{a_i^F} \tag{17}$$

so that:

$$\theta_y = \mathbf{r}^{\mathbf{T}}\mathbf{X}; \quad \varepsilon f(\mathbf{r}^{\mathbf{T}}\mathbf{X}, \mathbf{r}^{\mathbf{T}}\dot{\mathbf{X}}) = -M_y(\theta_y, \dot{\theta}_y) \tag{18}$$

and $\mathbf{x}_0$, $\mathbf{y}_0$ are constant column vectors governing the magnitude and phase characteristics of the external forcing.

These aeroelastic equations will now be solved numerically in the time domain and also using the averaging method as described in Section 3.

3. Averaged forms of the aeroelastic equations

In this section, the application of the method of averaging to the analysis of Eq. (15) is described. It will now be assumed that the motion of the control surface is described by the lowest two modes of the structure only, so that the aeroelastic equations become a two-dof second-order system. In the linear case, the flutter behaviour of a lifting surface alone may frequently be determined with acceptable accuracy by an analysis involving only these modes, particularly in the case where one mode involves predominantly bending and the other involves mostly torsion. In this investigation, the form of the aeroelastic equations used in the nonlinear analysis has therefore been kept as simple as possible, with the potential effect of higher modes on the stability domain analysis lying outside the scope of this study. In the case of unforced motion of this system ($\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{0}$), Zubov's method may be applied directly to Eq. (15). Averaging is therefore intended for use in the case of forced motion, and the range of forcing frequencies considered will be such that modes beyond the two retained in the formulation of the aeroelastic equations will have negligible effect. There are two distinct situations to be considered, namely the non-resonance and resonance conditions.

Consider the non-resonance case to begin with. Eq. (15) is first written as a four-dof first-order system as follows:

$$\bar{\mathbf{U}}\begin{pmatrix}\dot{\mathbf{X}}\\ \dot{\mathbf{Y}}\end{pmatrix} = \bar{\mathbf{V}}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} - \varepsilon\begin{pmatrix}\mathbf{0}\\ \mathbf{s}\end{pmatrix} f + \begin{pmatrix}\mathbf{0}\\ \mathbf{x}_0\end{pmatrix}\sin \omega_0 t + \begin{pmatrix}\mathbf{0}\\ \mathbf{y}_0\end{pmatrix}\cos \omega_0 t \tag{19}$$

where

$$\bar{\mathbf{U}} = \begin{pmatrix}-\mathbf{H} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}; \quad \bar{\mathbf{V}} = \begin{pmatrix}\mathbf{0} & -\mathbf{H}\\ -\mathbf{H} & -\mathbf{G}\end{pmatrix} \tag{20}$$

$\mathbf{I}$ denotes the 2×2 identity matrix and $\mathbf{Y} = \dot{\mathbf{X}}$. Let $\bar{\mathbf{N}}$ and $\bar{\mathbf{M}}$ denote the left hand and right hand matrices of eigenvectors that would arise in the eigenvalue problem that would occur when solving Eq. (19) for $f = 0$ and in the absence of forcing ($\mathbf{x}_0 = \mathbf{y}_0 = \mathbf{0}$). These will have the form:

$$\bar{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & \mathbf{M}^* \\ \mathbf{M}\boldsymbol{\mu} & \mathbf{M}^*\boldsymbol{\mu}^* \end{bmatrix}; \quad \bar{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & \boldsymbol{\mu}\mathbf{N} \\ \mathbf{N}^* & \boldsymbol{\mu}^*\mathbf{N}^* \end{bmatrix} \tag{21}$$

where $\boldsymbol{\mu} = \mathrm{diag}\{\mu_1, \mu_2\}$ with μ_1, μ_2 being the eigenvalues and * denoting the complex conjugate. The particular form of $\bar{\mathbf{N}}$ and $\bar{\mathbf{M}}$ arises because $\mathbf{Y} = \dot{\mathbf{X}}$. Thus if for an eigenvalue μ_i, where $\mathbf{X} = \mathbf{X}_0 \mathrm{e}^{\mu_i t}$, it follows that $\mathbf{Y} = \mathbf{X}_0 \mu_i \mathrm{e}^{\mu_i t}$. In this study, it is assumed that in the linearised aeroelastic equations considered μ_1, μ_2 will always possess both non-zero real and imaginary parts—i.e. divergence will not occur. Now define coordinates Q_1, Q_2, Q_3, Q_4 by the linear transformation:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} = \begin{bmatrix} \mathrm{Re}(\mathbf{M}) - \mathrm{Im}(\mathbf{M}) & \mathrm{Re}(\mathbf{M}) + \mathrm{Im}(\mathbf{M}) \\ \mathrm{Re}(\mathbf{M}\mu) - \mathrm{Im}(\mathbf{M}\mu) & \mathrm{Re}(\mathbf{M}\mu) + \mathrm{Im}(\mathbf{M}\mu) \end{bmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \tag{22}$$

where 'Re' and 'Im' denote the real and imaginary parts of the matrices concerned. With this transformation, Eq. (19) may now be written as:

$$\begin{pmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{Q}_3 \\ \dot{Q}_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \omega_1 & 0 \\ 0 & \lambda_2 & 0 & \omega_2 \\ -\omega_1 & 0 & \lambda_1 & 0 \\ 0 & -\omega_2 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} - \varepsilon \begin{pmatrix} \mathrm{Re}(\boldsymbol{\sigma}) + \mathrm{Im}(\boldsymbol{\sigma}) \\ \mathrm{Re}(\boldsymbol{\sigma}) - \mathrm{Im}(\boldsymbol{\sigma}) \end{pmatrix} f$$
$$+ \begin{pmatrix} \mathrm{Re}(\boldsymbol{\tau}_1) + \mathrm{Im}(\boldsymbol{\tau}_1) \\ \mathrm{Re}(\boldsymbol{\tau}_1) - \mathrm{Im}(\boldsymbol{\tau}_1) \end{pmatrix} \sin \omega_0 t + \begin{pmatrix} \mathrm{Re}(\boldsymbol{\tau}_2) + \mathrm{Im}(\boldsymbol{\tau}_2) \\ \mathrm{Re}(\boldsymbol{\tau}_2) - \mathrm{Im}(\boldsymbol{\tau}_2) \end{pmatrix} \cos \omega_0 t \tag{23}$$

where $\boldsymbol{\sigma} = \boldsymbol{\mu}\mathbf{N}\mathbf{s}$, $\boldsymbol{\tau}_1 = \boldsymbol{\mu}\mathbf{N}\mathbf{x}_0$, $\boldsymbol{\tau}_2 = \boldsymbol{\mu}\mathbf{N}\mathbf{y}_0$ and $\mu_k = \lambda_k + \mathrm{i}\omega_k$. It should be noted that ω_1 and ω_2 are the frequencies of the aeroelastic modes and are distinct from the lifting surface structure natural frequencies ω_1^F and ω_2^F.

Now make the following definitions:

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{\sigma}) + \mathrm{Im}(\boldsymbol{\sigma}) \\ \mathrm{Re}(\boldsymbol{\sigma}) - \mathrm{Im}(\boldsymbol{\sigma}) \end{pmatrix}; \quad \boldsymbol{\delta}_1 = \begin{pmatrix} \mathrm{Re}(\boldsymbol{\tau}_1) + \mathrm{Im}(\boldsymbol{\tau}_1) \\ \mathrm{Re}(\boldsymbol{\tau}_1) - \mathrm{Im}(\boldsymbol{\tau}_1) \end{pmatrix}; \quad \boldsymbol{\delta}_2 = \begin{pmatrix} \mathrm{Re}(\boldsymbol{\tau}_2) + \mathrm{Im}(\boldsymbol{\tau}_2) \\ \mathrm{Re}(\boldsymbol{\tau}_2) - \mathrm{Im}(\boldsymbol{\tau}_2) \end{pmatrix} \tag{24}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \omega_1 & 0 \\ 0 & \lambda_2 & 0 & \omega_2 \\ -\omega_1 & 0 & \lambda_1 & 0 \\ 0 & -\omega_2 & 0 & \lambda_2 \end{pmatrix} \tag{25}$$

and define a new vector of variables $\mathbf{Q}'$ such that:

$$\mathbf{Q} = \mathbf{Q}' + \mathbf{K}_1 \sin \omega_0 t + \mathbf{K}_2 \cos \omega_0 t \tag{26}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4)^{\mathrm{T}}$ and $\mathbf{K}_1$ and $\mathbf{K}_2$ are column vectors whose values are yet to be fixed. Eq. (26) is now substituted into Eq. (23) and $\mathbf{K}_1$ and $\mathbf{K}_2$ are chosen so as to remove all terms in $\sin \omega_0 t$ and $\cos \omega_0 t$ other than those that occur in the nonlinear function f. Thus $\mathbf{K}_1$ and $\mathbf{K}_2$ are then given by:

$$\mathbf{K}_1 = \frac{1}{\omega_0}(\boldsymbol{\Lambda}\mathbf{K}_2 + \boldsymbol{\delta}_2); \quad \mathbf{K}_2 = (-\omega_0^2 \mathbf{I} - \boldsymbol{\Lambda}^2)^{-1}(\boldsymbol{\Lambda}\boldsymbol{\delta}_2 + \omega_0 \boldsymbol{\delta}_1) \tag{27}$$

and Eq. (23) is reduced to:

$$\dot{\mathbf{Q}}' = \mathbf{\Lambda}\mathbf{Q}' - \varepsilon\boldsymbol{\gamma}f \tag{28}$$

making use of the expressions for $\boldsymbol{\gamma}$ given in Eq. (24) and $\mathbf{\Lambda}$ in Eq. (25). The method of averaging assumes that the nonlinearities in the system result in an oscillatory solution with slowly varying amplitude and phase. For the situation to be considered, it may be assumed that ω_1–$\omega_2 = O(\varepsilon)$ and $\lambda_1 = \lambda_2 = O(\varepsilon)$. Let the displacements Q'_1, Q'_2, Q'_3, Q'_4 now be transformed into a new set of displacements a_1, a_2, a_3, a_4 as follows:

$$\begin{pmatrix} Q'_1 \\ Q'_2 \\ Q'_3 \\ Q'_4 \end{pmatrix} = \begin{bmatrix} \sin\omega_1 t & 0 & \cos\omega_1 t & 0 \\ 0 & \sin\omega_1 t & 0 & \cos\omega_1 t \\ \cos\omega_1 t & 0 & -\sin\omega_1 t & 0 \\ 0 & \cos\omega_1 t & 0 & -\sin\omega_1 t \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \tag{29}$$

where a_1, a_2, a_3, a_4 will be assumed to be slowly varying. In terms of these new displacements, Eq. (28) may now be written as:

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \omega_1-\omega_2 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & \omega_2-\omega_1 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} - \varepsilon \begin{pmatrix} \gamma_1\sin\omega_1 t + \gamma_3\cos\omega_1 t \\ \gamma_2\sin\omega_1 t + \gamma_4\cos\omega_1 t \\ \gamma_1\cos\omega_1 t - \gamma_3\sin\omega_1 t \\ \gamma_2\cos\omega_1 t - \gamma_4\sin\omega_1 t \end{pmatrix} f \tag{30}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^{\mathrm{T}}$.

Consider now the nonlinear function f. In this application, f is a function of $\mathbf{r}^{\mathrm{T}}\mathbf{X}$. Substituting for $\mathbf{X}$ in terms of $\mathbf{Q}$ using Eq. (22), and in terms of $\mathbf{Q}'$ using Eq. (26), and ultimately in terms of a_1, a_2, a_3, a_4 using Eq. (29) results in:

$$\mathbf{r}^{\mathrm{T}}\mathbf{X} = \alpha_0\sin\omega_0 t + \beta_0\cos\omega_0 t + \alpha_1\sin\omega_1 t + \beta_1\cos\omega_1 t \tag{31}$$

where

$$\alpha_0 = [\mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) - \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\ \ \mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) + \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})]\mathbf{K}_1$$

$$\beta_0 = [\mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) - \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\ \ \mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) + \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})]\mathbf{K}_2$$

$$\alpha_1 = [\mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) - \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\ \ \mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) + \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})] \begin{pmatrix} a_1 \\ a_2 \\ -a_3 \\ -a_4 \end{pmatrix}$$

$$\beta_1 = [\mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) - \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\ \ \mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M}) + \mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})] \begin{pmatrix} a_3 \\ a_4 \\ a_1 \\ a_2 \end{pmatrix} \tag{32}$$

Noting that the non-resonance case is being considered, so that ω_1–$\omega_0 = O(1)$, the averaged form of Eq. (30) is now obtained by writing $\phi_0 = \omega_0 t$, $\phi_1 = \omega_1 t$ and then performing a double integration of Eq. (30) with respect to ϕ_0, ϕ_1 over the range 0–2π. The averaged form of the equations may then be written as:

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \omega_1-\omega_2 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & \omega_2-\omega_1 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} - \varepsilon \begin{pmatrix} \gamma_1 I_s + \gamma_3 I_c \\ \gamma_2 I_s + \gamma_4 I_c \\ \gamma_1 I_c - \gamma_3 I_s \\ \gamma_2 I_c - \gamma_4 I_s \end{pmatrix} \tag{33}$$

where

$$I_s = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f \sin\phi_1 \mathrm{d}\phi_0 \mathrm{d}\phi_1; \quad I_c = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f \cos\phi_1 \,\mathrm{d}\phi_0\,\mathrm{d}\phi_1 \tag{34}$$

For the particular case of a cubic hardening nonlinearity, f may be written as:

$$f(\mathbf{r}^{\mathrm{T}}\mathbf{X}) = K(\mathbf{r}^{\mathrm{T}}\mathbf{X})^3 \tag{35}$$

so that I_s and I_c will be given by:

$$I_s = \frac{3}{8}K\alpha_1(2\alpha_0^2 + 2\beta_0^2 + \alpha_1^2 + \beta_1^2); \quad I_c = \frac{3}{8}K\beta_1(2\alpha_0^2 + 2\beta_0^2 + \alpha_1^2 + \beta_1^2) \tag{36}$$

Consider now the resonance case. Eq. (15) are first transformed by defining the new variable $\mathbf{Z}$ such that:

$$\mathbf{Z} = \mathbf{PX} = \begin{pmatrix} r_1 & r_2 \\ \frac{-s_2}{|\mathbf{s}|} & \frac{s_1}{|\mathbf{s}|} \end{pmatrix}\mathbf{X} \tag{37}$$

where $\mathbf{r} = (r_1, r_2)^{\mathrm{T}}$; $\mathbf{s} = (s_1, s_2)^{\mathrm{T}}$. Eq. (15) may then be written as:

$$\ddot{\mathbf{Z}} + \mathbf{A}\dot{\mathbf{Z}} + \mathbf{BZ} + \varepsilon\begin{pmatrix} c \\ 0 \end{pmatrix} f(Z_1) = \mathbf{p}_0 \sin\omega_0 t + \mathbf{q}_0 \cos\omega_0 t \tag{38}$$

where

$$\mathbf{A} = \mathbf{PGP}^{-1}; \quad \mathbf{B} = \mathbf{PHP}^{-1}; \quad \mathbf{p}_0 = \mathbf{Px}_0; \quad \mathbf{q}_0 = \mathbf{Py}_0; \quad c = \mathbf{r}^{\mathrm{T}}\mathbf{s} \tag{39}$$

and $\mathbf{Z} = (Z_1, Z_2)^{\mathrm{T}}$. Now apply the invertible van der Pol transform. Thus:

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \dot{Z}_1 \\ \dot{Z}_2 \end{pmatrix} = \begin{bmatrix} \sin\omega_0 t & 0 & \cos\omega_0 t & 0 \\ 0 & \sin\omega_0 t & 0 & \cos\omega_0 t \\ \cos\omega_0 t & 0 & -\sin\omega_0 t & 0 \\ 0 & \cos\omega_0 t & 0 & -\sin\omega_0 t \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \tag{40}$$

where a_1, a_2, a_3, a_4 are slowly varying terms. The nonlinear term f is then written as a Fourier series expansion:

$$f(Z_1) = f(a_1 \sin\omega_0 t + a_3 \sin\omega_0 t) = J_s \sin\omega_0 t + J_c \cos\omega_0 t + \cdots \tag{41}$$

Eq. (41) is then substituted into Eq. (38), terms in $\cos\omega_0 t$ and $\sin\omega_0 t$ are equated with the higher harmonics being neglected. This results in the following set of autonomous equations for a_1, a_2, a_3, a_4:

$$\begin{pmatrix} 0 & -\mathbf{I}\omega_0 \\ \mathbf{I}\omega_0 & 0 \end{pmatrix}\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \\ \dot{a}_4 \end{pmatrix} + \begin{pmatrix} \mathbf{B} - \omega_0^2\mathbf{I} & -\omega_0\mathbf{A} \\ \omega_0\mathbf{A} & \mathbf{B} - \omega_0^2\mathbf{I} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \varepsilon c\begin{pmatrix} J_s \\ 0 \\ J_c \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{q}_0 \end{pmatrix} \tag{42}$$

For the particular case of a cubic hardening nonlinearity, where f is given by Eq. (35):

$$J_s = \frac{3}{4}Ka_1(a_1^2 + a_3^2); \quad J_c = \frac{3}{4}Ka_3(a_1^2 + a_3^2); \tag{43}$$

Eq. (33) for the non-resonance case and Eq. (42) for the resonance case will be used when considering the forced system.

4. Zubov's method for stability domain analysis

Zubov's method for the determination of the domain of stability of a stable equilibrium point of a system of autonomous nonlinear ordinary differential equations as it will be applied in this paper will now be discussed.

For an n dof autonomous nonlinear system given by:

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n) \tag{44}$$

for which $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ is an asymptotically stable equilibrium point, the domain of attraction of this point may be determined through obtaining a solution V to the partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i = -\phi(x_1, x_2, \ldots, x_n)(1 - V) \tag{45}$$

where $V(0,0,\ldots,0) = 0$ and:

$$\frac{\partial V}{\partial x_1}(0, \ldots, 0) = \frac{\partial V}{\partial x_2}(0, \ldots, 0) = \cdots = \frac{\partial V}{\partial x_n}(0, \ldots, 0) = 0 \tag{46}$$

where in Eq. (45), ϕ is a positive definite function. The function V satisfying Eq. (45), (46) is then a Lyapunov function determining the asymptotic stability of the equilibrium point $(0,0,\ldots,0)$ [25,26]. Furthermore, if a point $(x_1, x_2, \ldots, x_n)$ lies in the domain of stability of $(0,0,\ldots,0)$, then:

$$0 \leqslant V(x_1, x_2, \ldots, x_n) < 1 \tag{47}$$

and the boundary of the domain of stability is given by:

$$V(x_1, x_2, \ldots, x_n) = 1 \tag{48}$$

In practice exact solutions to Eq. (45) can be seldom found. However a unique series solution may be found in the form:

$$V(x_1, x_2, \ldots, x_n) = V_2(x_1, x_2, \ldots, x_n) + V_3(x_1, x_2, \ldots, x_n) + \ldots \tag{49}$$

where V_n denotes a homogeneous polynomial of degree n in $x_1, x_2, \ldots, x_n$. Substituting this form for V in Eq. (45) and equating coefficients successively for $V_2, V_3, \ldots$ enables the coefficients to be obtained by solving a series of linear simultaneous equations. Suppose that $V^{(N)}$ denotes the series expansion for V, truncated at degree N. Consider now the set $W^{(N)}$ of points $(x_1, x_2, \ldots, x_n)$ other than $(0,0,\ldots,0)$ for which:

$$\dot{V}^{(N)} = \sum_{i=1}^{n} \frac{\partial V^{(N)}}{\partial x_i} f_i = 0 \tag{50}$$

Then if $C^{(N)}$ is the minimum value of $V^{(N)}$ for $(x_1, x_2, \ldots, x_\mathrm{n})$ in $W^{(N)}$, then the boundary defined by:

$$V^{(N)}(x_1, x_2, \ldots, x_n) = C^{(N)} \tag{51}$$

will lie wholly within the domain of attraction of $(0,0,\ldots,0)$ [26]. This then provides the method for approximating a domain of attraction of an attractor.

A number of papers illustrate the application of this method [26,35]. It is well established that although taking increasing numbers of terms in the power series (49) leads to increasing accuracy of the stability domain predictions, this does not necessarily occur in a monotone manner, and consequently, a large number of terms may be needed to improve upon an initial estimate using only quadratic terms. However, an alternative approach is to define another function $v(x_1, x_2, \ldots, x_n)$ by the relation:

$$V = 1 - \mathrm{e}^{-v} \tag{52}$$

v then satisfies the partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i = -\phi(x_1, x_2, \ldots, x_n) \tag{53}$$

Eq. (53) is now solved by obtaining v as a power series solution, and the domain of attraction boundary estimate may then be found noting that $\dot{V} = 0$ if and only if $\dot{v} = 0$, and a maximum value of v will correspond to a minimum of V. There is experience to suggest that the use of the function v requires fewer terms in the power series solutions [35] and this approach will be adopted in the subsequent analysis. One limitation on the practical application of this approach has been the computational complexity that may arise from many dof.

However, with the use of computer algebra to construct the linear simultaneous equations determining the coefficients in the polynomial expansion of $V^{(N)}$ or $v^{(N)}$ this limitation may be alleviated so that the analysis of the fourth-order system of aeroelastic equations becomes very practical. Having obtained $V^{(N)}$ and $C^{(N)}$ (or the equivalents if working in terms of v), the estimated domain of attraction is fully defined for all initial conditions.

Finding solutions to the constrained optimisation problem defined by Eqs. (50) and (51) is achieved by using a sequential quadratic programming method. The starting solution may be taken as a point close to the stable equilibrium point. An alternative approach is to start close to an unstable equilibrium point, if there is one, as it will lie on the boundary of the domain of attraction of the stable equilibrium point.

The function ϕ must be positive definite, but other than this can be quite arbitrary. As has been done in other studies (e.g. Ref. [27]), in this paper, it will be chosen to be given by:

$$\phi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 \tag{54}$$

5. Results

In this section a number of examples of the use of Zubov's method for predictions of the domains of attraction of attractors for an all-moving control surface in supersonic flow, whose aeroelastic behaviour is governed by Eq. (15) are given, for free and forced motion. All examples consider a control surface for which full details are given in Appendix A. Given there are the geometric characteristics together with the natural modes and associated generalised masses and stiffnesses in the case when the root attachment torsional stiffness is zero. This modal data was derived from a finite element model of the lifting surface. For the determination of the aerodynamic loadings on the control surface, thickness effects were neglected. Flight is assumed to take place under sea level international standard atmosphere conditions.

These examples consider the case where the control surface possesses a cubic hardening nonlinearity in the root torsional dof. Thus the moment M_y in Eq. (2) may be written:

$$M_y = -K\mathbf{r}^{\mathrm{T}}\mathbf{X} - \varepsilon K(\mathbf{r}^{\mathrm{T}}\mathbf{X})^3 \tag{55}$$

where K is a linear torsional spring constant. In formulating the aeroelastic equations, the linear terms arising from Eq. (55) were incorporated into the matrix $\mathbf{H}$. Thus the function f could be written:

$$f(\mathbf{r}^{\mathrm{T}}\mathbf{X}) = K(\mathbf{r}^{\mathrm{T}}\mathbf{X})^3 \tag{56}$$

In this example K was taken as 75.9 Nm/rad. The flutter speed of the linear system ($\varepsilon = 0$) was first determined and speed was then non-dimensionalised with respect to this. Additionally, a non-dimensional time $t' = \tilde{\omega}t$ was defined so that the flutter frequency of the linear system would be $O(1)$. To achieve this the value of $\tilde{\omega}$ was taken as 1800 rad/s. The matrices $\mathbf{G}$, $\mathbf{H}$ and vectors $\mathbf{r}$, $\mathbf{s}$ that define the aeroelastic equations for the control surface could then be evaluated, using the data of Appendix A, to be:

$$\mathbf{G} = \begin{bmatrix} 1.84329 \times 10^{-2} & 2.48859 \times 10^{-3} \\ 3.98055 \times 10^{-4} & 1.81111 \times 10^{-2} \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 0.16727V_\infty + 0.42110 & 0.16820V_\infty + 0.49896 \\ -0.24317V_\infty + 7.98092 \times 10^{-2} & 1.31621 - 0.25748V_\infty \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 1.0 \\ 1.18488 \end{bmatrix}; \quad s = \begin{bmatrix} 0.421104 \\ 7.98092 \times 10^{-2} \end{bmatrix} \tag{57}$$

where V_∞ is the non-dimensional speed defined as explained previously such that $V_\infty = 1.0$ is the flutter speed for the linear system. The matrix $\mathbf{G}$ gives the aerodynamic damping terms and is independent of speed. This arises because control surface thickness effects have been neglected in the aerodynamics. Thus the integrals I_{1s},

I_{2s}, I_{3s}, I_{4s} and I_{5s} are zero so that the coefficients I_{ij} in Eq. (15) may be readily shown to be independent of speed.

Prior to the domain of stability investigations, a study was carried out to investigate the nonlinear aeroelastic behaviour of the system in the absence of forcing for $\varepsilon > 0$ in terms of the existence of possible limit cycles and their stability. The limit cycles were determined by using the averaged equations (33). However, in order to carry out this analysis, the equations were re-written in terms of the following transformed variables:

$$a_1 = A_1 \cos \phi_1; \quad a_2 = A_2 \cos \phi_2; \quad a_3 = A_1 \sin \phi_1; \quad a_4 = A_2 \sin \phi_2 \tag{58}$$

Eq. (33) may then be rewritten as:

$$\dot{A}_1 = \lambda_1 A_1 - \frac{\varepsilon\sqrt{2}}{2\pi}\int_0^{2\pi}(\sigma_{1R} \sin \psi_1 - \sigma_{1I} \cos \psi_1) f \, \mathrm{d}\psi_1$$

$$\dot{A}_2 = \lambda_2 A_2 - \frac{\varepsilon\sqrt{2}}{2\pi}\int_0^{2\pi}(\sigma_{2R} \sin(\theta+\psi_1) - \sigma_{2I} \cos(\theta+\psi_1)) f \, \mathrm{d}\psi_1$$

$$\dot{\theta} = \omega_2 - \omega_1 - \frac{\varepsilon\sqrt{2}}{2\pi A_2}\int_0^{2\pi}(\sigma_{2R} \cos(\theta+\psi_1) + \sigma_{2I} \sin(\theta+\psi_1)) f \, \mathrm{d}\psi_1 + \frac{\varepsilon\sqrt{2}}{2\pi A_1}\int_0^{2\pi}(\sigma_{1R} \cos \psi_1 + \sigma_{1I} \sin \psi_1) f \, \mathrm{d}\psi_1 \tag{59}$$

where $\theta = \psi_2 - \psi_1$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^{\mathrm{T}}$ and the subscripts 'R' and 'I' refer to the real and imaginary parts of σ_1, σ_2. $\mathbf{r}^{\mathrm{T}}\mathbf{X}$ may be rewritten in terms of A_1, A_2, ψ_2, ψ_1 as:

$$\mathbf{r}^{\mathrm{T}}\mathbf{X} = \sqrt{2}\,\mathrm{Re}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\begin{pmatrix} \sin \psi_1 \\ \sin(\psi_1+\theta) \end{pmatrix} + \sqrt{2}\,\mathrm{Im}(\mathbf{r}^{\mathrm{T}}\mathbf{M})\begin{pmatrix} \cos \psi_1 \\ \cos(\psi_1+\theta) \end{pmatrix} + \alpha_0 \sin \omega_0 t + \beta_0 \cos \omega_0 t \tag{60}$$

Limit cycles of Eq. (15) for zero forcing (so that $\alpha_0 = 0$, $\beta_0 = 0$) then correspond to non-zero equilibrium points (A_1^0, A_2^0, θ^0) of Eq. (59). A stability analysis of Eq. (59) for small perturbations about an equilibrium point will readily identify whether a limit cycle is stable or not.

For the system considered, Fig. 4 shows the variation of limit cycle amplitude with flight speed plotted in terms of $\sqrt{\varepsilon}|\mathbf{r}^{\mathrm{T}}\mathbf{X}|$ against V_∞. Stability analysis confirmed that the limit cycles were unstable and the results were checked using numerical integration in the time domain. Thus $V_\infty = 1.0$ is the speed at which sub-critical Hopf bifurcation occurs for the nonlinear system, with the equilibrium point O being stable for $V_\infty < 1.0$. Below $V_\infty = 0.81$, the unstable limit cycle vanishes, and the only attractor is the point O. Thus there are two flight regimes which will be of interest. In Regime I, for which $0 < V_\infty < 0.81$, the system possesses a sole stable equilibrium point at O. In Regime II, for which $0.81 < V_\infty < 1.0$, the system possesses a stable point at O together with the an unstable limit cycle. This is illustrated in Fig. 4.

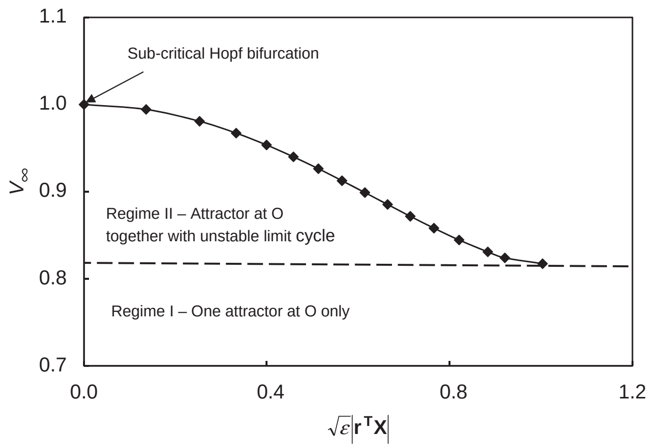

Fig. 4. Variation of limit cycle amplitude with non-dimensionalised speed V_∞.

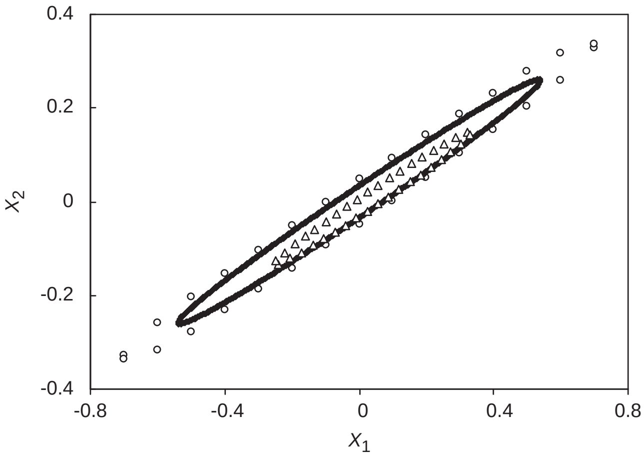

Fig. 5. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, Δ Zubov method $N = 2$, ♦ Zubov method $N = 4$.

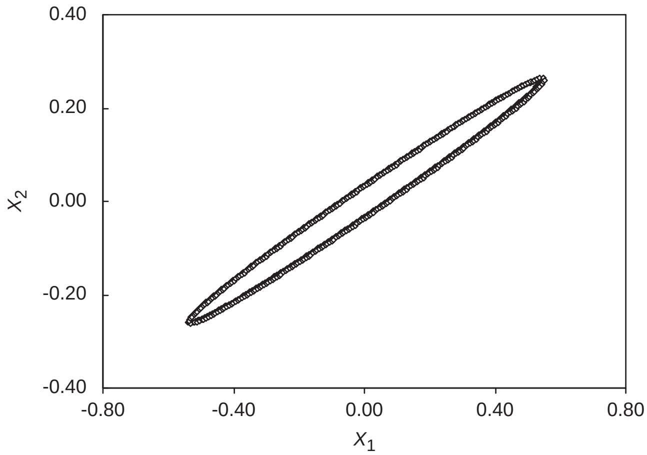

Fig. 6. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; ♦ Zubov method $N = 4$, Δ Zubov method $N = 6$.

For the initial study, $\varepsilon = 0.025$ and $V_\infty = 0.99431$; thus the nonlinearity is weak and the air-flow speed is very close to the linear flutter speed. The choice of value for ε then implied that $\mathbf{r}^{\mathbf{T}}\mathbf{X}$ should be regarded as being proportional to, rather than equal to, the root torsion angle θ_y. In the absence of forcing, Zubov's method may be applied to the time domain form of the aeroelastic Eq. (15) directly. This was done for power series expansions of the Lyapunov function up to order $N = 2$, 4 and 6. The nature of the nonlinear equations of motion is such that there are no terms of odd order in the expansions.

Comparisons of the domain of attraction of O are shown in Fig. 5 for $\dot{X}_1^{(0)} = 0$ and $\dot{X}_2^{(0)} = 0$. In the legend 'Time Domain Prediction' refers to numerical integration of Eq. (15). It is clear that $N = 4$ gives a much less conservative prediction of the domain of attraction than $N = 2$. Fig. 6 shows comparisons for $N = 4$ and $N = 6$ which indicates that little, if any, improvement is achieved by including the extra terms.

The stability domains of attractors of the aeroelastic equations in the presence of excitation were now investigated. This required the use of the averaged forms Eqs. (33) and (42) of the aeroelastic equations in order to work with autonomous equations as required in the Zubov analysis. The natural frequencies for the fin with the chosen airspeed were $\omega_1 = 0.937020$ and $\omega_2 = 0.979064$, so that it was the case that $\omega_1 - \omega_2 = O(\varepsilon)$. A preliminary first study was carried out using Eq. (33) in the absence of external excitation. Fig. 7 shows comparisons of the domain of attraction of O for $\dot{X}_1^{(0)} = 0$ and $\dot{X}_2^{(0)} = 0$ and Zubov analysis for

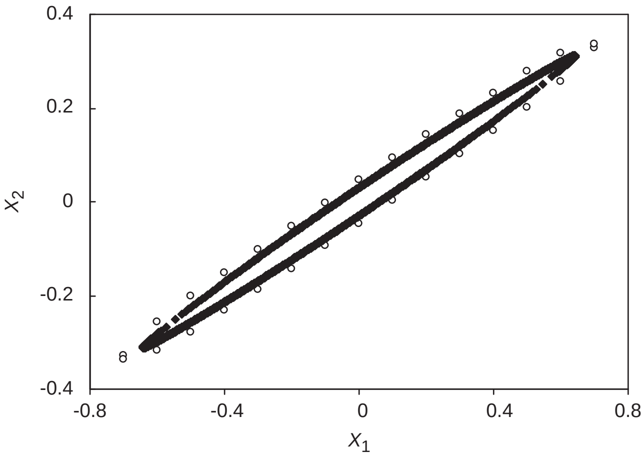

Fig. 7. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with averaged equations.

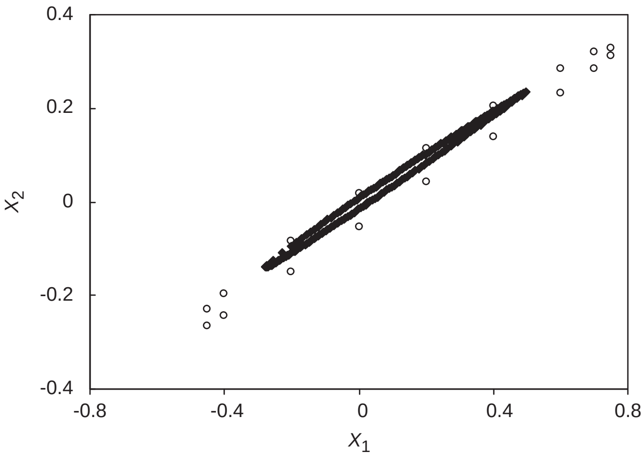

Fig. 8. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, initial conditions $\dot{X}_1^{(0)} = -0.25$, $\dot{X}_2^{(0)} = -0.5$; o time domain prediction, ♦ Zubov method with averaged equations.

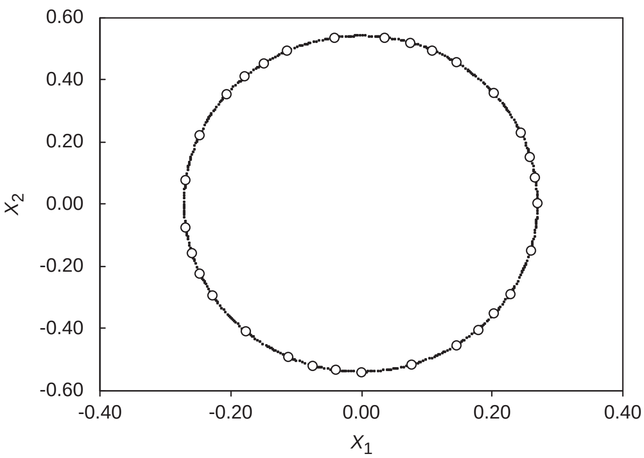

Fig. 9. Attractor predictions for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.75,0.75)^{\mathrm{T}}$, $\omega_0 = 2.0$; - - - time domain, o averaging.

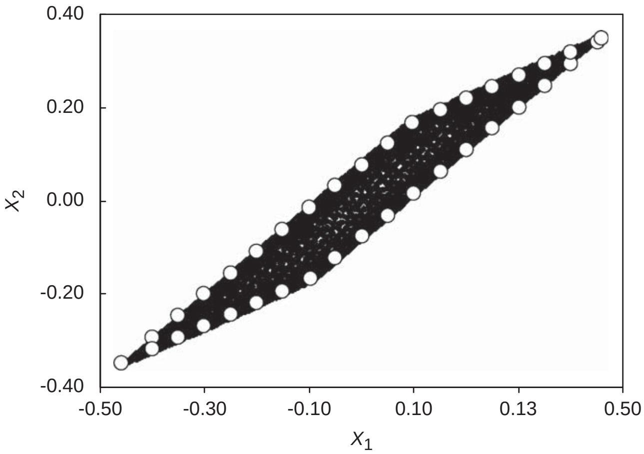

Fig. 10. Multi-periodic solution predictions for $\varepsilon = 0.025$, $V_\infty = 0.999431$; $\mathbf{y}_0 = (0.75{,}0.75)^T$; $\omega_0 = 2.0$; - - - time domain; o averaging.

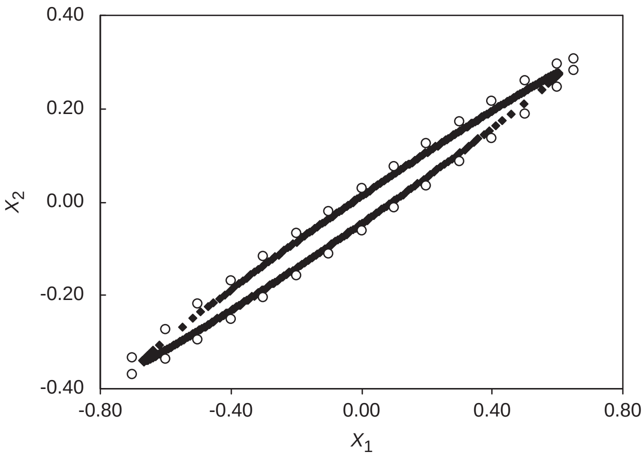

Fig. 11. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.10{,}0.10)^T$, $\omega_0 = 2.0$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with averaged equations.

the case $N = 4$. It is interesting to note that the Zubov domain of attraction estimate in this case is better than that obtained from working directly from the original form of the aeroelastic equations (15). As a further example, Fig. 8 shows domains of attraction predictions for $\dot{X}_1^{(0)} = -0.25$ and $\dot{X}_2^{(0)} = -0.5$.

The effect of forcing was now considered. The first studies were carried out for non-resonance cases, and thus Eq. (33) was used. Figs. 9 and 10 show how the attractors of the system are modified in the presence of excitation in the case where $\omega_0 = 2.0$ and $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.75{,}0.75)^T$. Time domain analysis readily reveals the existence of a periodic attractor which replaces the point attractor. This is shown in Fig. 9 together with the averaging method prediction. The attractors are found in the averaging method by setting $\dot{a}_1 = \dot{a}_2 = \dot{a}_3 = \dot{a}_4 = 0$ in Eq. (33). A multi-periodic solution may also be found in time domain analysis by a trial and error process to locate initial conditions resulting in trajectories very close to the solution. This multi-periodic solution may also be identified using the averaging method in the form of Eq. (59). The projection of the solution in the X_1–X_2 plane is shown in Fig. 10, and consists of a set of points whose closure will be a parallelogram as shown. The boundary of this parallelogram as predicted by the averaging method is also shown. Domain of attraction estimates were now determined for a number of cases. The results of these studies are shown in Figs. 11 and 12 for the cases $\omega_0 = 2.0$ and excitation levels of $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.10{,}0.10)^T$

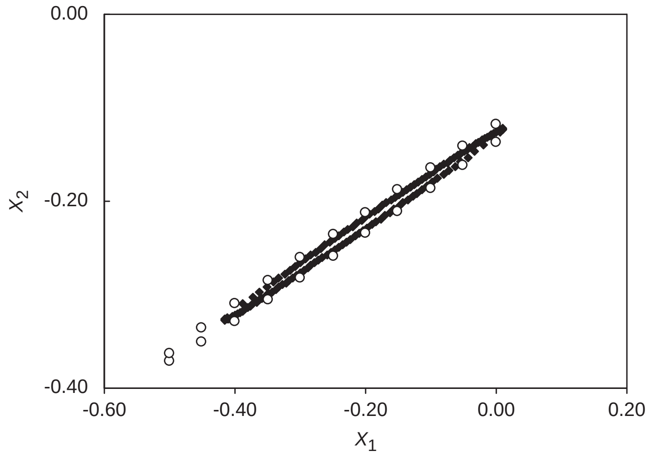

Fig. 12. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.75,0.75)^T$, $\omega_0 = 2.0$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with averaged equations.

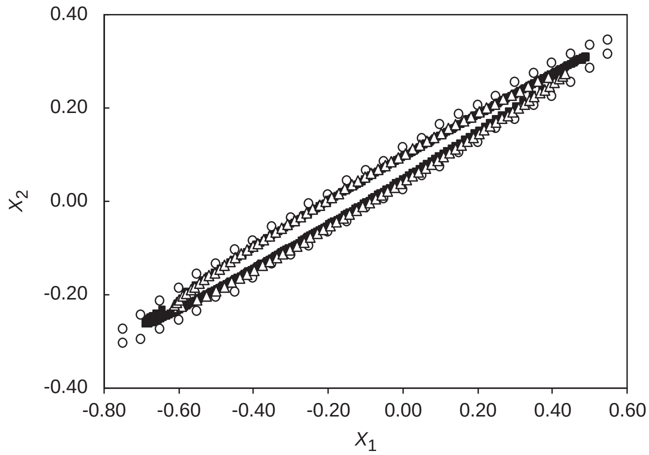

Fig. 13. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.02,0.02)^T$, $\omega_0 = 0.88$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with non-resonance averaged equations, △ Zubov method with resonance averaged equations.

and $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.75,0.75)^T$, respectively, for $\dot{X}_1^{(0)} = 0$ and $\dot{X}_2^{(0)} = 0$. In these and all subsequent cases, the Lyapunov function has been determined up to terms in $N = 4$. The resulting domain of attraction estimates agree well with time domain predictions.

A study for the resonance case was now carried out where $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.02,0.02)^T$, and forcing frequencies ω_0 were of the order 0.9. In all cases, domains of attraction were determined for $\dot{X}_1^{(0)} = 0$ and $\dot{X}_2^{(0)} = 0$. In this study, both the resonance (Eq. (42)) and non-resonance form (Eq. (33)) of the averaging equations were used. Figs. 13–15 show domain of attraction predictions for three ω_0 values close to 0.9. Figs. 13 and 14 show stability domain predictions for ω_0 values of 0.88 and 0.89, respectively, and it may be seen that close agreement between predictions using the two averaging method formulations and fair agreement with time domain predictions is obtained. For $\omega_0 = 0.9$, Fig. 15 shows that Zubov analysis using both averaging formulations give very different but conservative stability predictions in both cases indicating that the stability domain was close to disappearing. However, it should be noted that with an increase in ω_0 to 0.91, time domain stability domain analysis predicts the disappearance of the domain of attraction.

All the studies discussed hitherto have been for an air speed in Regime II, where $0.81 < V_\infty < 1.0$. A study was now carried out for $V_\infty = 0.766687$, which lies in Regime I. In this case, in the absence of excitation, the

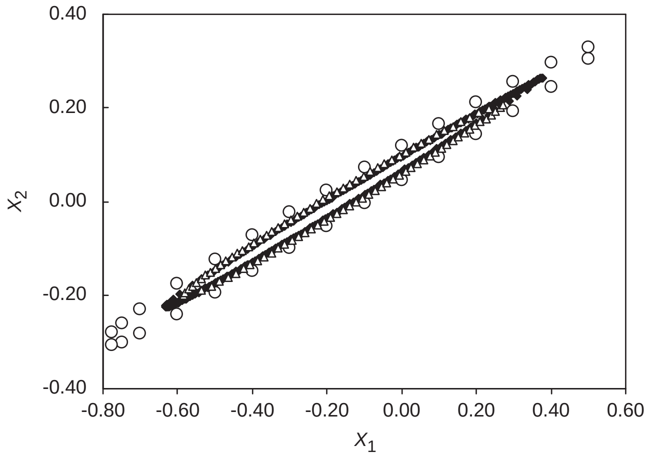

Fig. 14. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.02,0.02)^T$, $\omega_0 = 0.89$, initial conditions $\dot{X}_1^{(0)} = 0.0$; $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with non-resonance averaged equations, Δ Zubov method with resonance averaged equations.

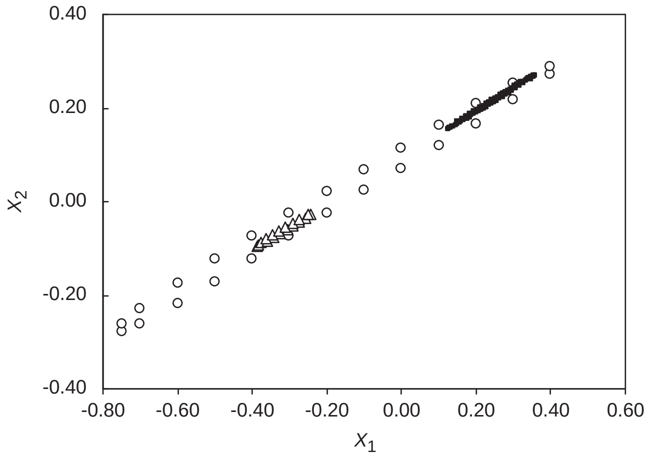

Fig. 15. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.999431$, $\mathbf{y}_0 = (0.02,0.02)^T$, $\omega_0 = 0.90$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; o time domain prediction, ♦ Zubov method with non-resonance averaged equations, △ Zubov method with resonance averaged equations.

system has one attractor only. In the presence of excitation, a typical frequency response is as shown in Fig. 16, for which the excitation level is given by $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.3,0.3)^T$. Predictions have been made using time domain analysis and averaging using the van der Pol form of averaging (Eq. (42)) by setting $\dot{a}_1 = \dot{a}_2 = \dot{a}_3 = \dot{a}_4 = 0$. For frequencies in the region 0.85–0.95, there are two periodic attractors together with one unstable periodic solution. The two periodic attractors will possess domains of attraction and these will now be investigated using the Zubov method in conjunction with the resonance form of the averaging equations Eq. (42). In this case, the equilibrium points, which correspond to the periodic solutions, are non-zero, and thus the polynomial expansion for the Lyapunov function has to include odd terms. The domain of attraction of the periodic attractor with the smaller amplitude will be estimated. The expansion was taken up to $N = 4$ in all these cases. Figs. 17–19 show stability domain predictions for $\omega_0 = 0.95$ and increasing levels of excitation for $\dot{X}_1^{(0)} = 0$ and $\dot{X}_2^{(0)} = 0$. Thus what is shown is a two-dimensional section through the full domain of attraction, which is contained in a four-dimensional space. The shaded regions indicate the domain of

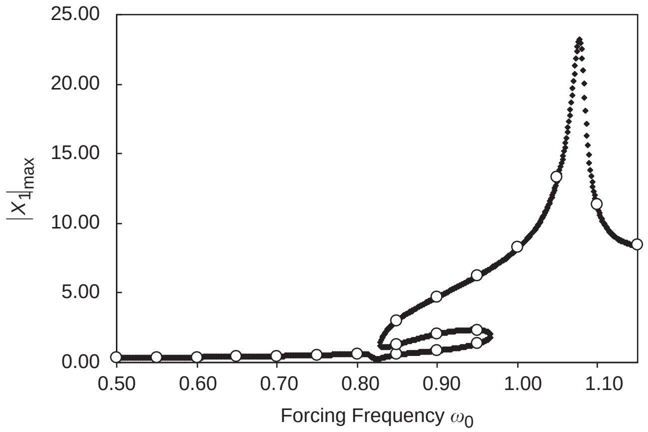

Fig. 16. Frequency response for $\varepsilon = 0.025$, $V_\infty = 0.766687$, $\mathbf{y}_0 = (0.3,0.3)^T$; o time domain prediction, ♦ averaging method.

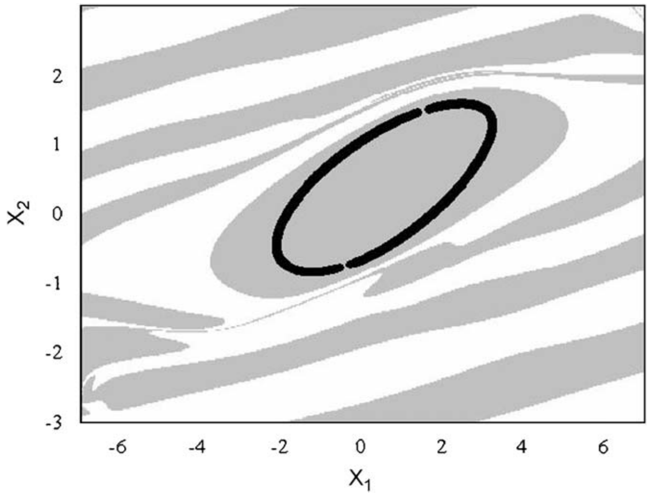

Fig. 17. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.766687$, $\mathbf{y}_0 = (0.1, 0.1)^T$, $\omega_0 = 0.95$, initial conditions $\dot{X}_1^{(0)} = 0.0, \dot{X}_2^{(0)} = 0.0$; ▢ time domain prediction, ● Zubov method prediction of boundary.

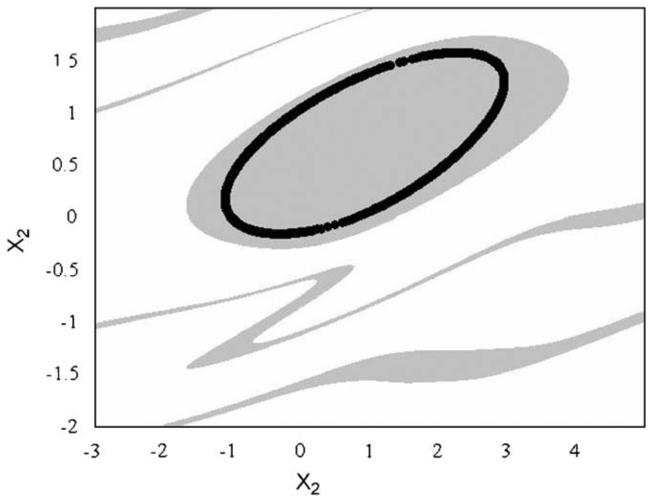

Fig. 18. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.766687$, $\mathbf{y}_0 = (0.2,\ 0.2)^T$, $\omega_0 = 0.95$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; ▢ time domain prediction, ● Zubov method prediction of boundary.

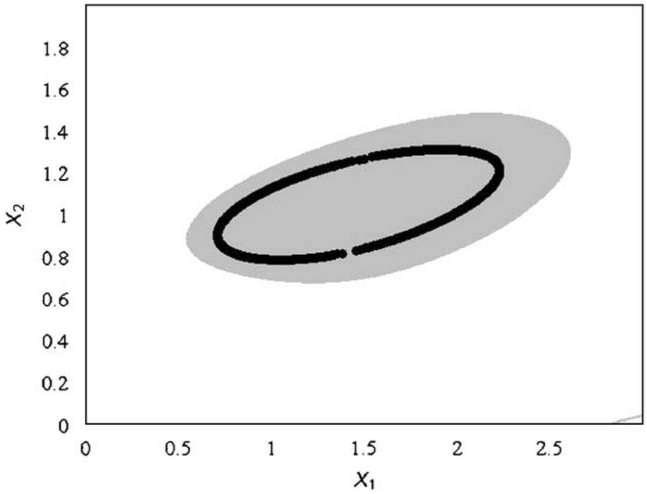

Fig. 19. Domain of attraction estimates for $\varepsilon = 0.025$, $V_\infty = 0.766687$, $\mathbf{y}_0 = (0.3,\ 0.3)^T$, $\omega_0 = 0.95$, initial conditions $\dot{X}_1^{(0)} = 0.0$, $\dot{X}_2^{(0)} = 0.0$; ☐ time domain prediction, ● Zubov method prediction of boundary.

attraction for the attractor with smaller amplitude, while the white regions indicate the domain of attraction of the attractor with the larger amplitude as predicted by time domain analysis. The curve in black indicates the boundary of the domain of attraction for the attractor with small amplitude as predicted by the Zubov method. Fig. 17 shows comparisons of time domain and Zubov-based predictions for $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.1,0.1)^T$. It will be seen that the stability domain as predicted by time domain studies has a very complex geometry, consisting of a central domain surrounded by a series of bands. When the excitation level is increased to $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.2,0.2)^T$, the central region persists, although it is reduced in size and the surrounding bands are much thinner. For a further increase in the excitation level to $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{y}_0 = (0.3,0.3)^T$ the central region has reduced further still, and the surrounding bands have largely disappeared. In all cases, the Zubov method gives not unreasonable, and in all cases conservative, estimates of the central region of the stability domains. The bands which form part of the small amplitude attractor domain of attraction are not captured by the Zubov's method. However, from a practical point of view, if one wished to specify ranges of initial conditions which would assure a return to the attractor, the most straightforward way to do this would be based on the central domain of the attractor. From this perspective, the fact that the Zubov approach does not capture the full complexity of the domains of attraction need not be a major drawback.

6. Concluding remarks

In this paper, the aeroelastic behaviour at high Mach numbers of an all-moving control surface with a nonlinearity in the root support has been investigated. The particular situation investigated is that where a stable equilibrium point, corresponding to zero displacement of the structure, together with an unstable limit cycle typically arising from a sub-critical Hopf bifurcation results from the presence of the nonlinearity so that the stable equilibrium point will then possess a domain of attraction. The dynamic aeroelastic response is also of interest. When sinusoidal forcing is applied, the stable equilibrium point may then be replaced by a periodic attractor, and the limit cycle by a multi-periodic solution. With or without this forcing, there is an attractor which will possess a domain of attraction. The problem of estimating these domains of attraction has been tackled using Zubov's method. In the absence of forcing, the method was applied directly to the aeroelastic equations, while for the forced system, the method of averaging was applied to approximate the aeroelastic equations by an autonomous system. The behaviour of the system was also investigated for flight speeds below the threshold where the unstable limit cycle of the unforced system disappears. In this regime, the forced nonlinear system may still possess multiple attractors, and the domain of attraction of the attractors for this situation was investigated also, again using an averaged form of the aeroelastic equations.

The partial differential equation arising in Zubov's method was solved by a series expansion approach which was made viable for the system of nonlinear aeroelastic equations by the use of computer algebra. For the cases considered, it was possible to obtain a fair approximation to the domain of attraction with a fourth order series expansion, and going to sixth order gave little improvement. Two forms of averaging were applied when analysing the forced response of the system. It was shown that for the case where the air speed was such that the unforced system possessed an unstable limit cycle, the non-resonance form of the averaging method resulted in stability domain predictions at least as good as those obtained from the resonance form of the averaging equations even if this form of the equations was strictly more appropriate. For air speeds below this threshold, and for forcing frequencies where multiple attractors existed, the method was shown to be capable of giving a good indication of the extent of the central region of the complex attractors that occurred. Although their full complexity is not captured by the Zubov method, from a practical point of view, this need not be a major drawback since if one wished to specify ranges of initial conditions which would assure a return to the attractor, the most straightforward way to do this would be based on the central domain of the attractor.

The examples considered indicate that, in general, the Zubov method gives a good indication of the domain of attraction of an attractor. The method is highly efficient in that once the function $V^{(N)}$, together with $C^{(N)}$, have been found, the estimated domain of attraction within the four-dimensional phase space is fully determined. The conservative nature of the estimates is also an advantage from a practical point of view. Through the use of computer algebra, the Zubov method may be readily implemented for multi-dof systems. Although this investigation focuses on an aeroelastic system, the form of equations considered is such that the approach would be relevant to many other engineering applications.

Appendix A. Numerical data for aeroelastic models

The lifting surface considered in these studies had the geometric characteristics shown in Fig. A1. Thickness effects are neglected.

The mode shapes, together with generalised mass and stiffness data are given in Tables A1 and A2.

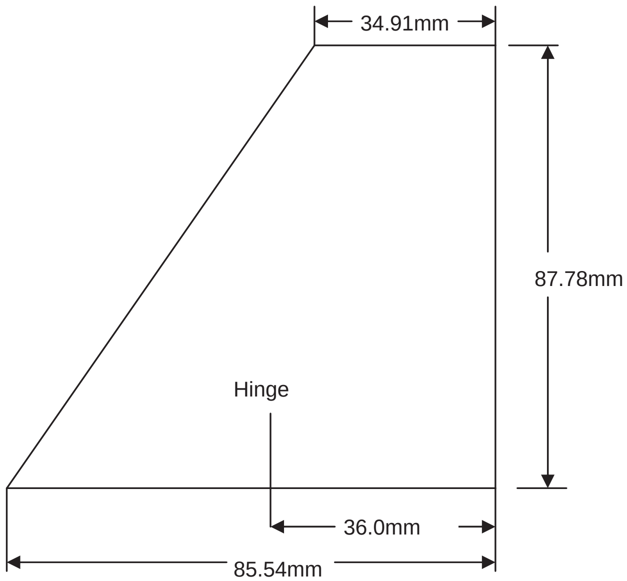

Fig. A1. Lifting surface geometry.

Table A1
Modal data.

Strip numbers	$2b_s$ (mm)	$2b_s x_{os}$ (mm)	ϕ^F_{1s}	ϕ^F_{2s}	ψ^F_{1ys}	ψ^F_{2ys}
1	80.4738	46.6290	-2.125213×10^{-3}	3.124895×10^{-3}	1.000000	1.158449
2	70.3494	40.7550	-6.375600×10^{-3}	1.528817×10^{-2}	1.000000	1.133542
3	60.2250	34.8810	-1.062600×10^{-2}	3.386578×10^{-2}	1.000000	1.078578
4	50.1006	29.0070	-1.487640×10^{-2}	5.785579×10^{-2}	1.000000	1.041359
5	39.9762	23.1330	-1.912680×10^{-2}	8.590975×10^{-2}	1.000000	1.005500

Table A2
Generalised mass and stiffness data.

Mode number i	a^F_i	e^F_i	ψ^F_{iy0}
1	5.562880×10^{-5}	0.0	1.000000
2	3.477841×10^{-4}	1.589684×10^{3}	1.184876

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